Minimal Model Program

Learning Seminar.

Week 41.

Sarkisov Program.

Maximal Mori fiber spaces.

The Sarkisov Program:

Theorem: (Z, Φ) is a klt proj pair with $K_Z + \Phi$ pseff. Assume $Z \overset{\phi_1}{\dashrightarrow} X_1$ and $Z \overset{\phi_2}{\dashrightarrow} X_2$ are two minimal models. $\Delta_i := \phi_{i*}\Phi$. Then $X_1 \dashrightarrow X_2$ is a composition of $(K_{X_1} + \Delta_1)$-flops.

Q: (Z, Φ) is a klt proj pair with $K_Z + \Phi$ not pseff.

$$\begin{tikzcd} Z \arrow[r, dashed, "\phi_1"] & X \arrow[d] \\ & S \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \arrow[r, dashed, "\phi_2"] & Y \arrow[d] \\ & T \end{tikzcd}$$

be two MFS.

How can we factor $X \dashrightarrow Y$?

Def: Two MFS are Sarkisov related if both of them are obtained by a MMP from Z.

Thm 1.1: $\phi: X \longrightarrow S$ and $\psi: Y \longrightarrow T$ MFS with $\mathbb{Q}$-fact sing. Then X and Y are birational $\Longleftrightarrow$ they are connected by a sequence of Sarkisov links.

Motivation:

$Cr(n) = Bir(\mathbb{P}^n)$.

$\varphi \in Bir(\mathbb{P}^2) =: Cr(2)$.

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed, "\varphi"] \arrow[d] & \mathbb{P}^2 \arrow[d] \\ pt & pt. \end{tikzcd}$$

$$\varphi = \underbrace{\psi_1 \circ \dots \circ \psi_s}_{\text{simpler elements in } Cr(2).}$$

Noether – Castelnuovo: $Cr(2)$ is generated by $\underline{PGL(3, \mathbb{C})}$ + $\underline{\text{Cremona trans.}}$

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ [x:y:z] &\longmapsto [x^{-1}:y^{-1}:z^{-1}] = [yz:xz:xy]. \end{aligned}$$

1983: Gizatullin: Gave a description of the relations

1927: Hudson: $Cr(3)$ is not generated by elements of bounded degree

Q: What are the generators of $Cr(3)$?

Corti proved that the Sarkisov program works in dim 3.

$$\begin{tikzcd} \mathcal{L}_X \subseteq X \arrow[r, dashed, "\sigma"] \arrow[d, "\sigma_1"'] & Y \supseteq \mathcal{L} \text{ linear system.} \\ \mathcal{L}_{X_1} \subseteq X_1 & \end{tikzcd}$$

How the sing of $\mathcal{L}_X$ & $\mathcal{L}_{X_1}$ compare?

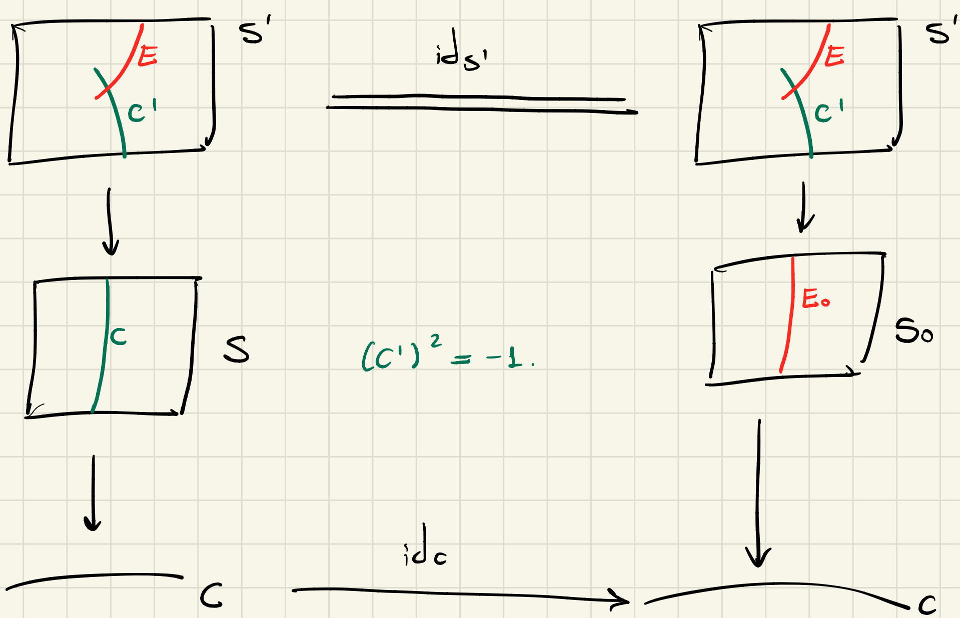
canonical threshold + pseff thresholds.

Sarkisov Links:

$$\begin{tikzcd}
\mathbb{F}_1 \arrow[r, "id_{\mathbb{F}_1}"] \arrow[d] & \mathbb{F}_1 \arrow[d, "\psi"] \\
\mathbb{P}^2 \arrow[d, "\phi"'] & \mathbb{P}^1 \arrow[dl] \\
pt &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X' \arrow[r, dashed, "flops"] \arrow[d] & Y \arrow[d, "\psi"] \\
X \arrow[d, "\phi"'] & T \arrow[dl] \\
S &
\end{tikzcd}$$

Link of type I.

Links of type III are the reflection of type I links



$(C')^2 = -1.$

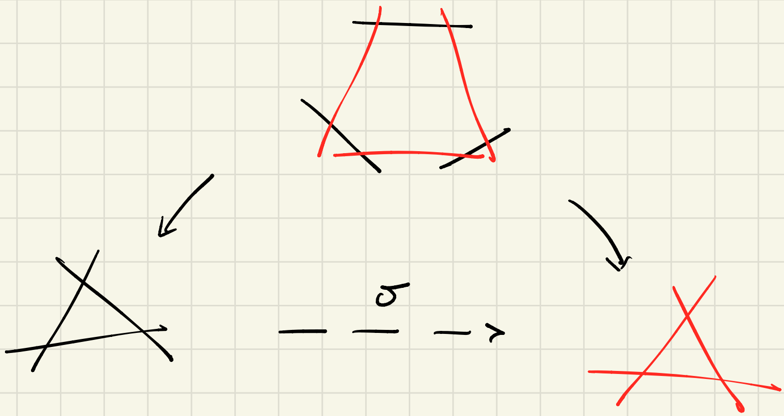
Link of type II:

$$
\begin{tikzcd}
X' \arrow[r, dashed, "\text{flops}"] \arrow[d] & Y' \arrow[d] \\
X \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\
S \arrow[r, equal] & T
\end{tikzcd}
$$

Link of type IV:

$$
\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, equal] \arrow[d, "\phi"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\psi"] \\
\mathbb{P}^1 \arrow[dr] & \mathbb{P}^1 \arrow[d] \\
 & pt
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\
S \arrow[dr] & & T \arrow[dl] \\
 & R &
\end{tikzcd}
$$

Sarkisov of the Cremona:



Maximal Mori fiber space structure:

Def: X klt & projective is said to be a max MFS if X admits a MFS structure and every K_X-negative extremal curve of X induces a MFS.

conjectural

$$\begin{tikzcd} X \arrow[d, dashed] & K_X \text{ pseff} \\ X' & \text{minimal model, } K_X \text{ semiample} \end{tikzcd}$$

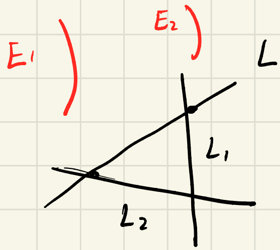
conj; due to BCHM.

$$\begin{tikzcd} X \arrow[d, dashed] & K_X \text{ not pseff} \\ X' \arrow[d, dashed] & \text{has a MFS.} \\ X'' & \text{all its negative rays induce MFS.} \end{tikzcd}$$

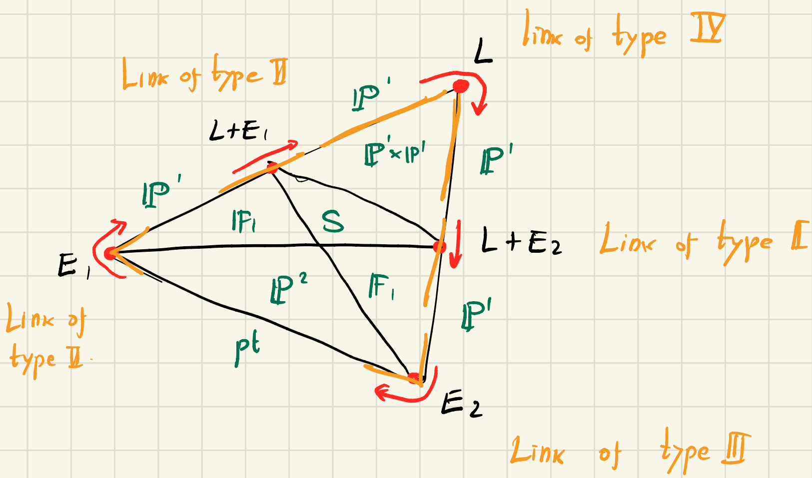
Q: Let X be proj klt with K_X not pseff.
Is X birational to a <u>maximal</u> MFS?

Idea of the proof:

$S = Bl_{p,q}\, \mathbb{P}^2$



Cone of effective divisors is generated by L, E_1 & E_2.



Ambient space Z, $V \subseteq NS(Z)$ linear subs def over $\mathbb{Q}$ finite dim.

$$\mathcal{L}_A(V) = \{ \Theta = A + B \in V_A \mid K_Z + \Theta \text{ is lc and } B_Z \geq 0 \}$$

$$\mathcal{E}_A(V) = \{ \Theta \in \mathcal{L}_A(V) \mid K_Z + \Theta \text{ pseff} \}.$$

$f : Z \dashrightarrow X$ we define

$$\mathcal{A}_{A,f}(V) := \{ \Theta \in \mathcal{E}_A(V) \mid f \text{ is ample model for } (Z, \Theta) \}$$

$$C_{A,f} := \overline{\mathcal{A}_{A,f}(V)}$$

Theorem 3.3: There are finitely many $f_i : Z_i \dashrightarrow X$ satisfying the following:

(1) • $\mathcal{A}_i := \mathcal{A}_{A,f_i}$ is a partition of $\mathcal{E}_A(V)$
• $\mathcal{A}_i$ is a finite union of interiors of rat polytopes.
• If f_i is birational, then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is rational

BCHM.

(2) $\mathcal{A}_j \cap \mathcal{C}_i \neq \phi$, then there is a contraction morphism
$f_{i,j} : X_i \longrightarrow X_j$, $f_j = f_{i,j} \circ f_i$.

bpf thm.

Assume V spans the Neron - Severi

(3) Connected component $\mathcal{C}$ of $\mathcal{C}_i$ that intersects $\mathcal{L}_A(V)$.

Then TFAE:

$\Theta \in \text{int}(\mathcal{C} \cap \mathcal{A}_i)$, $f : Z \dashrightarrow X$ log terminal models.
$f = f_j$, $\mathcal{A}_i \cap \mathcal{A}_j \neq \phi$, $\mathcal{A}_i = \mathcal{A}_j$. $i = j$.

i) $\mathcal{C}$ spans V.
ii) If $\Theta \in \mathcal{A}_i \cap \mathcal{C}$, then f_i is a log terminal model of $K_Z + \Theta$
iii) f_i is birational and X_i is $\mathbb{Q}$-factorial.

log terminal models are bir

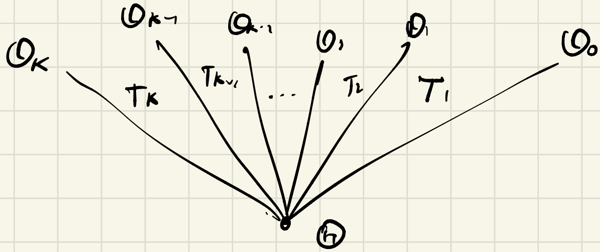
(4) $\mathcal{C}_i$ spans V and Θ is general in $\mathcal{A}_i \cap \mathcal{C}$ and lies in the interior of $\mathcal{L}_A(V)$. Then the relative Picard of $f_{i,j} : X_i \longrightarrow X_j$ equals $\dim(\mathcal{C}_i) - \dim(\mathcal{C}_i \cap \mathcal{C}_j)$

Cone Thm

Notation: $\Theta = A + B$ is in the boundary of $\mathcal{E}_A(V)$ and in the interior of $\mathcal{L}_A(V)$.

$T_1, \dots, T_k$ polytopes $\mathcal{C}_i$ of dim 2 containing Θ.

$$\mathcal{O}_0 = T_1 \cap \partial \mathcal{E}_A(V)$$
$$\mathcal{O}_k = T_k \cap \partial \mathcal{E}_A(V)$$
$$\mathcal{O}_i = T_i \cap T_{i+1}$$



$f_i : Z \dashrightarrow X_i$ ample model ass to T_i.

$g_i : Z \dashrightarrow S_i$ ample model ass to $\mathcal{O}_i$.

$f = f_1 : Z \dashrightarrow X$, $\quad g = f_k : Z \dashrightarrow Y$, $\quad X' = X_2$, $Y' = X_{k-1}$

$\phi : X \longrightarrow S = S_0$, $\quad \psi : Y \longrightarrow T = S_k$

$Z \dashrightarrow R$ ample model of $K_Z + \Theta$.

Thm 3.7: $K_Z + \Phi$ klt and $\Theta - \psi$ ample.
Then ϕ and ψ are two MFS which are outputs of $(K_Z + \Phi)$-MMP which are by a Sarkisov link if Θ is contained in more than two polytopes.

Proof: Commutative heptagon

$$\begin{tikzcd}
& X' \arrow[rr, dashed] \arrow[d, dashed, "p"] & & Y' \arrow[d, dashed, "q"] \\
Z \arrow[r, dashed, "f"] \arrow[rrr, dashed, bend right=15] & X \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\
& S \arrow[dr, "s"'] & & T \arrow[dl, "t"] \\
& & R & &
\end{tikzcd}$$

$\Delta = f_* \Theta$

$K_X + \Delta$ is numerically trivial over R

Both ϕ and ψ are MFS + outcomes of $(K_Z + \Phi)$-MMP's.

(3.3) we can prove $\rho(X'/R) = 2$, $\rho(Y'/R) = 2$

- p is a divisorial contr + $s = \mathrm{id}$, or
- p is a flop + s is not the identity.

- q is a divisorial contr + $r = \mathrm{id}$, or
- q is a flop + r is not the identity. $\square$

Lemma 4.1: $\phi: X \longrightarrow S$ and $\psi: Y \longrightarrow T$

are Sarkisov related MFS associated to (X, Δ) and (Y, Γ)

We may find $f: Z \dashrightarrow X$ and $g: Z \dashrightarrow Y$,

(Z, Φ), klt, A ample on Z, V two dimensional

in $\mathrm{WDiv}_{\mathbb{R}}(Z)$ such that:

(1) if $\Theta \in \mathcal{L}_A(V)$, then $\Theta - \Phi$ ample,

(2) $\mathcal{A}_{A, \phi \circ f}$ and $\mathcal{A}_{A, \psi \circ g}$ are not in $\partial \mathcal{L}_A(V)$,

(3) V satisfies (1-4) of Thm 3.3.

(4) $\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ are 2-dim, and

(5) $\mathcal{C}_{A, \phi \circ f}$ and $\mathcal{C}_{A, \psi \circ g}$ are 1-dim

Proof of 4.1:

- Replacing with a log resolution, we may assume (Z, ϕ) is log smooth and f and g are morphism.

- Add divisors to the boundary, so the components of the boundary span the Neron - Severi:

$A, H_1, \ldots, H_k$ ample s.t $H_1, \ldots, H_k$ generate $NS(Z)$

$H = A + H_1 + \ldots + H_k$.

C in S ample, D in T ample

$-(K_X + \Delta) + \phi^* C$ and $-(K_Y + \Gamma) + \psi^* D$ are ample

Pick $0 < \delta < 1$ such that.

$-(K_X + \Delta + \delta f_* H) + \phi^* C$ and $-(K_Y + \Gamma + \delta g_* H) + \psi^* D$ ample

$K_Z + \Phi + \delta H$ is f-neg and g-neg. Assume $\delta = 1$.

Pick $\Phi_0 \leq \Phi$ s.t $A + \underbrace{(\Phi_0 - \Phi}_{\text{small enough}})$ ample

$$\left.\begin{aligned} &-(K_X + f_* \Phi_0 + f_* H) + \phi^* C \\ &-(K_X + g_* \Phi_0 + g_* H) + \psi^* D \end{aligned}\right\} \text{ample}$$

$K_Z + \Phi_0 + H$ is f-neg and g-neg.

Pick $F_1 \geq 0$, $G_1 \geq 0$ $\mathbb{Q}$-general

$K_Z + \Phi_0 + H + F + G$ klt, where $F = f^* F_1$ & $G = g^* G_1$

$$V_0 = \Phi_0 + \langle H_1, \dots, H_k, F, G \rangle.$$

$$\Theta - \Phi = \underbrace{\underbrace{(A + \Phi_0 - \Phi)}_{\text{ample}} + \underbrace{B - \Phi_0}_{\text{nef}}}_{\text{ample}}$$

$\Phi_0 + F + H \in \mathcal{A}_{A, \phi \circ f}(V_0)$ and.

$\Phi_0 + G + H \in \mathcal{A}_{A, \phi \circ g}(V_0)$,

V_0 satisfies (1-4) of (3.3).

Finally, we need to cut down the dimension of V_0. $\square$

Proof of 1.3: (Z, Φ), A and V as in 4.1.

$\Theta_0 \in \mathcal{A}_{A, \phi \circ f}(V)$, $\Theta_1 \in \mathcal{A}_{A, \psi \circ g}(V)$ in the int of $\mathcal{L}(X)$

Θ_t. there are finitely points Θ_i is contained in more than two polytopes $\mathcal{C}_{A, f_i}(V)$.

(3.7) implies that the corresponding $\sigma_i: X \dashrightarrow Y$ is composition of Sarkisov links. $\square$